

LOG CANONICAL THRESHOLDS OF DEL PEZZO SURFACES IN CHARACTERISTIC p

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ABSTRACT. The log canonical threshold for all smooth complex del Pezzo surfaces was computed by Cheltsov. His proof used connectivity of the non-klt locus and adjunction-type results, relying on vanishing theorems not known to be true in characteristic p . We compute the log canonical threshold of non-singular del Pezzos over an algebraically closed field. We give an algebraic proof of this adjunction-type for surfaces and avoid the connectivity result by classifying curves of small degree. As an application, non-singular del Pezzo surfaces over an algebraically closed field are K -semistable if their degree is lower than 3.

1. INTRODUCTION

The log canonical threshold (lct) of an algebraic variety¹ X , $\text{lct}(X)$, is a numerical invariant introduced by Shokurov in the setting of the Minimal Model Program. It related to some of the conjectures of this program such as the ACC conjecture, recently proved in [HMX12]. ACC implies termination of certain kind of flips.

Definitions 1.1. A **log pair** $(X, D = \sum d_i D_i)$ is a pair where X is a variety and $D \subset X$ an effective \mathbb{Q} -Cartier divisor.

A **resolution** of (X, D) is a proper and birational morphism $\sigma : Y \rightarrow X$ such that Y is non-singular, \tilde{D} , the strict transform of D , is a divisor with non-singular support and σ is an isomorphism when restricted to $Y \setminus (\text{Ex}(f))$.

A **log resolution** of (X, D) is a resolution such that $\sigma^{-1}(D)$ is a divisor with simple normal crossings (snc) and such that the exceptional locus of σ has pure codimension 1.

Log resolutions can be obtained from embedded resolutions. The latter are resolutions in which $\text{Supp}(D)$ is seen as a subvariety of X and X and D are resolved *at the same time*. Once an embedded resolution is found it can be easily modified into a log resolution.

Finding embedded resolutions or abstract resolutions of a pair (X, D) for X of dimension n over an algebraically closed field k is an open problem. Embedded resolutions exist when $\text{char}(k) = 0$ for all dimensions. For algebraically closed fields

Date: 04 September 2012.

2010 Mathematics Subject Classification. Primary 14J45; Secondary 14G17.

Key words and phrases. del Pezzo surfaces, log canonical thresholds, positive characteristic, K -stability, birational automorphisms, intersection of two quadrics.

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¹Unless otherwise mentioned, all varieties in this paper are projective normal varieties over an algebraically closed field k of characteristic p , not necessarily 0.

of finite characteristic they exist when $n \leq 3$ thanks to a recent result by Cossart and Piltant [CP08],[CP09].

Assume X is \mathbb{Q} -factorial and let $\sigma: \tilde{X} \rightarrow X$ be a proper birational modification of the pair (X, D) . We may write

$$(1) \quad K_{\tilde{X}} + \tilde{D} + \sum a(F_j, X, D)F_j \equiv \sigma^*(K_X + D),$$

where the F_j are exceptional divisors and \tilde{D} the strict transform of D . We call $a(F_j, X, D)$, the **discrepancy of F_j with respect to (X, D)** and we often write a_j if no confusion is likely. Denote by $\text{Disc}(X, D) = \sup\{a(F_j, X, D)\}^2$, the **discrepancy** of the pair (X, D) where supremum is taken over all resolutions σ of (X, D) .

Definition 1.2. Let (X, D) be a log pair. We say (X, D) is **log canonical** if $\text{Disc}(X, D) \leq 1$.

We say (X, D) is **log canonical at $p \in X$** if $(U, D|_U)$ is log canonical for U a Zariski open set of p .

By [Kol97, Cor 3.13] it is enough to consider a log resolution to decide if (X, D) is log canonical.

Definition 1.3.

- (i) The **log canonical threshold of the pair (X, D)** is

$$\text{lct}(X, D) = \sup\{\lambda : (X, \lambda D) \text{ is log canonical}\}.$$

- (ii) The **local log canonical threshold** of the pair (X, D) is

$$\text{lct}_p(X, D) = \sup\{\lambda : (X, \lambda D) \text{ is log canonical at } p\}.$$

- (iii) The **(global) log canonical threshold** of X is

$$\text{lct}(X) = \sup\{\lambda : (X, \lambda D) \text{ is log canonical } \forall D \sim_{\mathbb{Q}} -K_X \text{ effective } \mathbb{Q}\text{-divisor}\}.$$

Obviously $\text{lct}(X, D) \leq \text{lct}_p(X, D)$ for all $p \in X$. It is not yet known whether $\text{lct}(X)$ is rational. The following conjecture generalises the one in [CPS10, Conj 1.4] for complex Fano varieties:

Conjecture 1.4. *Let X be a projective Fano variety over an algebraically closed field. Suppose X is \mathbb{Q} -factorial and has at most log terminal singularities. Then*

- (i) $\exists D \sim_{\mathbb{Q}} -K_X$, an effective \mathbb{Q} -divisor on X such that $\text{lct}(X) = \text{lct}(X, D)$.
- (ii) $\text{lct}(X)$ is a rational number.

Note that (i) implies (ii). This conjecture is not known to be true even for complex del Pezzo surfaces. However, there is strong evidence to support it. In fact, in the case of complex del Pezzo surfaces with Du Val singularities, D can be found in $|-mK_X|$ for $m = 1, 2$.

In [CS08, Thm. A.3], Demailly gives an elegant proof of the following well known result: the log canonical threshold of a polarised compact complex manifold (X, L) coincides with Tian's α -invariant $\alpha(X)$ introduced in [Tia87] when (X, L) is seen as a non-singular complex projective variety with positive first Chern Class. This is the same as a non-singular projective Fano variety. The main result in [Tia87] is

²Sometimes the a_j in (1) are defined with a minus sign and our supremum becomes an infimum (see [Kol97]).

that if $\alpha(X) > \frac{n}{n+1}$ where n is the dimension of X , then X can be equipped with a Kähler-Einstein (KE) metric.

Ten years later, Tian proved [Tia97] that the existence of a KE metric in a non-singular Fano variety is a sufficient condition for X to be analytically K -stable. The definition of K -stability is rather technical and it involves the use of *test configurations* so we refer the reader to [Oda].

There are several inequivalent concepts of stability and it is an open problem to prove that the existence of a KE metric is equivalent to some kind of algebro-geometric stability. A short and beautiful survey about these ideas, together with ideas to complete the proof can be found in [Don10]. A direct algebraic proof of the relation between log canonical thresholds and K -stability avoiding Kähler-Einstein metrics was recently published:

Theorem 1.5 (Odaka, Sano [OS12]). *Let X be a \mathbb{Q} -Fano variety of dimension n and suppose that $\text{lct}(X) > \frac{n}{n+1}$ (resp. $\text{lct}(X) \geq \frac{n}{n+1}$). Then, $(X, \mathcal{O}_X(-K_X))$ is K -stable (resp. K -semistable).*

Their proof uses resolution of singularities for dimension n , so it is valid in finite characteristic when $\dim(X) \leq 3$.

Although it is introduced above in the context of Kähler-Einstein metrics, K -stability is interesting in birational geometry on its own right. For instance, in [Oda], Odaka shows that given certain conditions if (X, L) is K -stable where L is an ample line bundle, then X has only semi-log canonical singularities (the proof is for $\text{char}(k) = 0$).

The purpose of this paper is to study the log canonical thresholds of del Pezzo Surfaces, proving the following result:

Theorem 1.6 (Main Theorem). *Let X be a nonsingular del Pezzo surface over an algebraically closed field k . Then:*

$$\omega = \text{lct}(X) = \begin{cases} 1 & \text{when } K_X^2 = 1 \text{ and } |-K_X| \text{ has no cuspidal curves} \\ 5/6 & \text{when } K_X^2 = 1 \text{ and } |-K_X| \text{ has a cuspidal curve} \\ 5/6 & \text{when } K_X^2 = 2 \text{ and } |-K_X| \text{ has no tacnodal curves} \\ 3/4 & \text{when } K_X^2 = 2 \text{ and } |-K_X| \text{ has a tacnodal curve} \\ 3/4 & \text{when } K_X^2 = 3 \text{ and } \forall C \in |-K_X|, C \text{ has no Eckard points} \\ 2/3 & \text{when } K_X^2 = 3 \text{ and } \exists C \in |-K_X| \text{ with an Eckard point} \\ 2/3 & \text{when } K_X^2 = 4 \\ 1/2 & \text{when } K_X^2 = 5, 6 \text{ or } X \cong \mathbb{P}^1 \times \mathbb{P}^1 \text{ } (K_X^2 = 8) \\ 1/3 & \text{when } K_X^2 = 7, 9 \text{ or } X \cong \mathbb{F}_1 \text{ } (K_X^2 = 8) \end{cases}$$

Since Theorem 1.5 works for surfaces, we see that the del Pezzo surfaces of degrees 1, 2 together with the one of degree 3 with no Eckard points are K -stable, and the del Pezzo surfaces of degrees 1 to 4 are K -semistable. Moreover we verify Conjecture 1.4 for del Pezzo surfaces. To the best of the author's knowledge, these are the first examples of K -stable varieties over fields of finite characteristic.

The first result of this type for del Pezzo was by Park [Par01]. He showed that if the pairs considered were (X, D) for D a divisor (i.e. with integer rather than rational coefficients), then the log canonical threshold had to be *at most* the one above. Although that paper assumes $k = \mathbb{C}$, the proof of this particular result is purely algebraic. We can therefore assume that $(X, \omega D)$ is log canonical for all effective $D \subset \text{Pic}(X)$, $D \sim -K_X$ for X a del Pezzo surface.

Theorem 1.6 was proved by Cheltsov when $k = \mathbb{C}$ in [Che08]. His proof is algebraic for degrees different than 2, 3, 4, where he uses results that require fields of characteristic 0 (see Lemma 1.7 below).

1.1. Organisation of the paper and techniques used. In section 2 we remind the reader the basic classification of del Pezzo surfaces. Furthermore, we introduce general results in log canonicity that we will use in the paper. In particular we give an algebraic proof of [Che08, Lem. 2.5] which is used when $K_X^2 = 2, 3$ (see our Lemma 2.5 (iii) below). With this modification Cheltsov's proof of Theorem 1.6 becomes characteristic free when $K_X^2 \neq 2, 4$. The case $K_X^2 = 2$ is done in section 4 modifying the $k = \mathbb{C}$ case to work also in low characteristics.

The main part of the paper (section 3) deals with the case $K_X^2 = 4$, the complete intersection of two quadrics in \mathbb{P}^4 . The method we use to find the threshold is somehow new. As usual, we first find the *worst* divisor or higher bound, i.e $D \in |-mK_X|$ such that $\text{lct}(X) \leq \text{lct}(X, \frac{1}{m}D)$. We should note that it is enough to take $m = 1$. Then we need to show that if $(X, \frac{2}{3}D)$ is not log canonical at $p \in X$ we get a contradiction. So far, this approach is standard. Our method is new on how to get the contradiction. We look for a divisor H depending on an arbitrary point at p and satisfying certain conditions (see Lemma 3.9). These conditions and the existence of H , together with the tools in section 2.1 will give the contradiction (section 3.4). The construction of H is done by case analysis in section 3.5 and it is combinatorial in nature.

It is worth mentioning the technique used by Cheltsov for $k = \mathbb{C}$ and $K_X^2 = 4$ and why it cannot be used in this case. His proof used the connectedness of the locus of log canonical singularities:

Lemma 1.7 ([Pro01] Cor. 2.3.3). *Let X, Z be normal complex varieties and $f : X \rightarrow Z$ be a contraction and D an effective \mathbb{Q} -divisor on X such that $K_X + D$ is \mathbb{Q} -Cartier. Assume $-(K_X + D)$ is f -nef and f -big. Then $\text{LCS}(X, D)$ is connected in a neighbourhood of f .*

The proof of this lemma uses Kawamata-Viehweg vanishing theorem for $k = \mathbb{C}$. In characteristic p there is a counter-example of Kawamata-Viehweg in dimension 3. Tanaka has recently proved a Nadel-type vanishing theorem for surfaces in finite characteristic (see [Tan12]) but it is not strong enough for Lemma 1.7. In the case of surfaces [Sho92] gave a proof of a similar result (whenever $K + D \equiv 0$) which is believed to work for finite characteristic. However his arguments are hard to verify and at some points they may be imprecise. For this reason when $K_X^2 = 4$ we find a completely different argument using the explicit knowledge of low-degree curves in del Pezzo surfaces.

The lemmas used in Section 2 (in particular Lemma 2.5 (iii)), together with the connectedness result are all that has ever been used to compute the log canonical thresholds of a surface. Should Shokurov's proof be carefully verified, most of the computations in the literature for different classes of complex surfaces would work over algebraically closed fields providing the arrangements of low degree curves do not change with the characteristic of the field.

ACKNOWLEDGMENTS

I would like to thank my advisor Ivan Cheltsov for introducing me to this problem as well as for his invaluable support and advice. I would also like to thank Yuji Sano for clarifications regarding [OS12].

2. BASIC TOOLS

2.1. Results in log canonicity. We start by introducing the locus of the singularities we are concerned with:

Definition 2.1. The **locus of log canonical singularities** of a log pair $(X, D = \sum d_i D_i)$ as in (1) is the closed set:

$$\text{LCS}(X, D) = \left(\bigcup_{d_i \geq 1} D_i \right) \cup \left(\bigcup_{a_j \geq 1} \sigma(F_j) \right) \subsetneq X.$$

This is called the *non-klt locus* in [Kol97].

Let S be a surface with canonical singularities. By [KM98, 4.11, 4.19] S is \mathbb{Q} -factorial. Let $\gamma : \tilde{S} \rightarrow S$ be a birational morphism and D a \mathbb{Q} -divisor in S with proper transform \tilde{D} . Then we can write (1) as

$$K_{\tilde{S}} + \tilde{D} + \sum_{i=1}^r a_i E_i \equiv \gamma^*(K_S + D),$$

where E_i are exceptional curves ($E_i \cong \mathbb{P}^1, E_i^2 < 0$) and a_i are rational numbers.

Lemma 2.2. *The log pair (S, D) is log canonical if and only if*

$$(2) \quad (\tilde{S}, \tilde{D} + \sum_{i=1}^r a_i E_i)$$

is log canonical.

2.1.1. Local properties.

Notation 2.3. We fix the following notation for the rest of the paper. We will be dealing with a pair (S, D) (or $(S, \omega D)$) where S is a surface which is non singular. Let $p \in S$ and $D \equiv -K_X$ be an effective \mathbb{Q} -divisor ($\omega \in \mathbb{Q} \cap [0, 1]$) such that (S, D) (respectively $(S, \omega D)$) is not log canonical at p .

Let C be an irreducible curve on the surface S . We may write $D = mC + \Omega$, where m is a non-negative rational number, and $\Omega = \sum a_i \Omega_i$ is an effective \mathbb{Q} -divisor such that $C \not\subset \text{Supp}(\Omega)$.

We will denote by $\sigma : \tilde{S} \rightarrow S$ the blow up at p with exceptional divisor E , and $q \in E$ a point at which the pair

$$(3) \quad (\tilde{S}, \tilde{D} + (\text{mult}_p D - 1)E) \quad (\text{or } (\tilde{S}, \omega \tilde{D} + (\omega \text{mult}_p D - 1)E))$$

is not log canonical at q . This point exists by Lemma 2.2.

Lemma 2.4 (Convexity). *Given S non-singular (at p), let D, B be effective \mathbb{Q} -divisors on S such that B is log canonical (at p) and D is not log canonical (at p). Then, $\forall \alpha \in [0, 1) \cap \mathbb{Q}$,*

$$D' = \frac{1}{1 - \alpha}(D - \alpha B)$$

is not log canonical (at p). Moreover if $D \equiv B$, then $D' \equiv D$ and we can choose α such that $\exists B_i$ irreducible curve ($p \in B_i$) in the support of B with $B_i \not\subset \text{Supp}(D')$.

Proof. See, for instance [Wil10, Lemma 5]. \square

Lemma 2.4 is true even if p is a canonical singularity.

Lemma 2.5. *For (S, D) , p and C as in Notation 2.3 the following are true:*

- (i) $\text{mult}_p D > 1$.
- (ii) If $C \subset \text{LCS}(S, D)$, then $m \geq 1$.
- (iii) If $m \leq 1$ and $p \in C$ with C non-singular at p , then:

$$C \cdot \Omega > 1 \quad (\text{Adjunction}).$$

Proof. Part (ii) is straight forward from the definition of $\text{LCS}(S, D)$. For (i) and (iii) consider $f: \tilde{S} \rightarrow S$, a log resolution of (S, D) , where the components of $f^{-1}(D)$ have simple normal crossings. By Lemma 2.2 since D is not log canonical, $\exists q \in f^{-1}(p)$ where (2) is not log canonical. We do induction on the number N of exceptional divisors of f .

Let us prove (i). In the initial step of induction, D is smooth at p , so we can assume $D = aD_1$ around p . Since (S, D) is not log canonical, $a > 1$, so $\text{mult}_p(D) = a > 1$. Assuming (i) for N , if f has $N+1$ exceptional divisors, blowing up p leaves the pair in (3) not log canonical and its log resolution can be achieved in N blow-ups. Hence for some $q \in E \cap \text{Supp}(\tilde{D})$

$$2\text{mult}_p(D) \geq \text{mult}_q(\tilde{D}) + \text{mult}_p(D) > 2,$$

by the inductive step. This proves (i).

For (iii), in the initial step of induction D already has snc, we show $(C \cdot \Omega) > 1$. Suppose it is not. Then, $\text{mult}_p \Omega \leq C \cdot \Omega \leq 1$. Since Ω is effective, for all i with $\text{mult}_p \Omega_i > 0$ we have $a_i \leq 1$. Since $m \leq 1$ and we have snc, then D is log canonical at p , a contradiction.

For the inductive step we consider that in (2) $r = 1$ and

$$(\tilde{S}, m\tilde{C} + \tilde{\Omega} + (\text{mult}_p D - 1)E)$$

is not log canonical at some $q \in E$ by (3). By the induction hypothesis

$$\tilde{C} \cdot (\tilde{\Omega} + (\text{mult}_p D - 1)E) > 1.$$

Since C is non-singular at p , we have

$$C \cdot \Omega - \text{mult}_p \Omega = \tilde{C} \cdot \tilde{\Omega} > 2 - \text{mult}_p D,$$

so $C \cdot \Omega > 2 - m \geq 1$. \square

Corollary 2.6. *If $(S, \omega D)$ is not log canonical at p and $q \in \tilde{C}$ then*

$$\tilde{C} \cdot \tilde{\Omega} > \frac{2}{\omega} - \text{mult}_p D.$$

Corollary 2.7. *If (S, D) (respectively $(S, \omega D)$) is not log canonical at p , then*

$$\text{mult}_q \tilde{D} + \text{mult}_p D > 2 \quad (\text{respectively } \text{mult}_q \tilde{D} + \text{mult}_p D > \frac{2}{\omega}).$$

2.2. del Pezzo surfaces. We recall some standard results of surfaces that we will use often.

Theorem 2.8 (Nakai-Moishezon criterion [Har77, V.I.10]). *A divisor D on a surface S is ample if and only if $D^2 > 0$ and $D \cdot C > 0$ for all irreducible curves C in S .*

This theorem suggests the following definitions:

Definition 2.9. A **del Pezzo surface** X over an algebraically closed field k is a non-singular surface whose anticanonical divisor, $-K_X$ is ample. Given any effective \mathbb{Q} -divisor $D \neq 0$, its **anticanonical degree** (or just degree) is the positive rational number defined by

$$\deg(D) = (-K_X) \cdot D.$$

If D is a divisor, $\deg(D)$ is a positive integer.

The **degree** of X is the positive integer

$$\deg(X) = (K_X)^2.$$

We will call effective divisors of degrees 1, 2, 3, ... lines, conics, cubics... respectively.

Theorem 2.10 ([Man86, Chapter IV, Theorem 24.3 (ii)]). *Let X be a del Pezzo surface. Then every irreducible curve with a negative self-intersection number is exceptional.*

Definition 2.11. A set of distinct points $\{p_1, \dots, p_r\}$ on \mathbb{P}_k^2 with $r \leq 8$ are in **general position** if no three of them lie on a line, no six of them lie on a conic and a cubic containing 7 points, one of them double, does not contain the eighth one.

We can characterise del Pezzo surfaces:

Theorem 2.12 ([Man86, Chapter IV, Theorems 24.3, 24.4, 26.2]). *Let X be a del Pezzo surface of degree d . Then $1 \leq d \leq 9$ and either $X = \mathbb{P}^1 \times \mathbb{P}^1$ ($\deg X = 8$) or X is a blow-up of \mathbb{P}^2 in $9 - d$ points in general position:*

$$\pi: X \rightarrow \mathbb{P}^2.$$

Conversely, any blow-up of \mathbb{P}^2 in $9 - d$ points in general position, for $1 \leq d \leq 9$ is a del Pezzo surface of degree d .

*We call the morphism π a **model** of X .*

There are further ways of classifying del Pezzo surfaces. For instance, $\deg(X) = 4$, if and only if X is the non-singular intersection of two quadrics in \mathbb{P}^4 . $\deg(X) = 3$ if and only if X is a non-singular cubic surface.

Theorem 2.12 implies that del Pezzo surfaces are rational. The following result applies:

Proposition 2.13. *For S a non-singular rational surface and $C \subset S$, a curve with arithmetic genus $p_a(C) = 0$, we have*

$$(4) \quad h^0(S, \mathcal{O}_S(C)) = (-K_S) \cdot C.$$

Proof. Let $C \in |C|$ be an effective divisor. By Serre Duality:

$$h^2(S, \mathcal{O}_S(C)) = h^0(S, \mathcal{O}_S(K_S - C)) = 0,$$

since S is rational. By the Riemann-Roch theorem:

$$h^0(S, \mathcal{O}_S(C)) \geq \frac{1}{2}C \cdot (C - K_S) + 1 = -K_S \cdot C + p_a(C),$$

where we use the genus formula. \square

3. DEL PEZZO SURFACE OF DEGREE 4

Let X be a del Pezzo surface of degree 4. In this section we prove $\text{lct}(X) = \frac{2}{3}$. Let $p \in X$ be some point. We will denote by $\sigma : \tilde{X} \rightarrow X$ be the blow-up at p , with exceptional divisor E . Let $q \in E$. First we describe the curves of low degree, forcing sometimes that they pass through p or that their strict transform σ passes through q . We also show how to choose an appropriate model for X so that these curves look reasonable. The proof comes right afterwards and uses the existence of certain effective divisors G and H depending on $p \in X$ and $q \in \tilde{X}$ and satisfying certain properties. G and H are constructed explicitly in section 3.5.

3.1. Catalogue of curves of low degree. Let $\pi : X \rightarrow \mathbb{P}^2$ be the blow-up at $p_1, \dots, p_5 \in \mathbb{P}^2$ in general position. Let E_1, \dots, E_5 be the exceptional divisors. Recall $-K_X \sim \pi^*(\mathcal{O}_{\mathbb{P}^2}(3)) - \sum_{i=1}^5 E_i$ and $E_i^2 = -1$.

Linear system \mathcal{LS}			$\deg C$	C^2	Fix p	Fix q	C'
$ E_i $			1	-1	N	N	E_i
$\mathcal{L}_{ij} = \pi^*(\mathcal{O}_{\mathbb{P}^2}(1)) - E_i - E_j $			1	-1	N	N	L_{ij}
$\mathcal{C}_0 = \pi^*(\mathcal{O}_{\mathbb{P}^2}(2)) - \sum_{i=1}^5 E_i $			1	-1	N	N	C_0
$\mathcal{B}_i = \pi^*(\mathcal{O}_{\mathbb{P}^2}(1)) - E_i $			2	0	Y	N	B_i
$\mathcal{H}_i = \pi^*(\mathcal{O}_{\mathbb{P}^2}(2)) - \sum_{\substack{j=1 \\ j \neq i}}^5 E_j $			2	0	Y	N	H_i
$\mathcal{Q}_i = \pi^*(\mathcal{O}_{\mathbb{P}^2}(3)) - E_i - \sum_{j=1}^5 E_j $			3	1	Y	Y	Q_i
$\mathcal{R} = \pi^*(\mathcal{O}_{\mathbb{P}^2}(1)) $			3	1	Y	Y	R
$\mathcal{R}_{ijk} = \pi^*(\mathcal{O}_{\mathbb{P}^2}(2)) - E_i - E_j - E_k $			3	1	Y	Y	R_{ijk}

TABLE 1. Catalogue of low degree curves in X .

Observe Table 1. In the first column we have defined certain complete linear systems \mathcal{LS} in X . Let $C \in \mathcal{LS}$ be any divisor. Its numerical properties $(C^2, \deg(C))$ are the same for any divisor in a given \mathcal{LS} and are easy to compute. We list them in the second and third columns of table 1. Note that, by the genus formula, $p_a(C) = 0$ in all cases.

If $\deg C = 2$, then by Proposition 2.13, $h^0(\mathcal{LS}) \geq 2$. Take $\mathcal{LS}' \subset \mathcal{LS}$ to be the sublinear system fixing p . Then $h^0(\mathcal{LS}') \geq 1$ and we can find a curve $C' \in \mathcal{LS}$ with $p \in C'$. The notation for each particular C' is in the last column of the table.

When the curve C' is irreducible, we can see it as the strict transform of an irreducible curve in \mathbb{P}^2 via the model π . For instance L_{ij} is the strict transform of the unique line through p_i and p_j . C_0 is the strict transform of the unique conic through all p_i . B_i is the strict transform of a line passing through p_i and H_i is the strict transform of a conic through all p_j but p_i . The cubics can be expressed similarly.

3.2. Lines and conics of X . Changing the blow up from \mathbb{P}^2 . In order to understand the geometry of X we need to understand which are its curves of low degree and how they intersect each other. We have seen some of these curves. We want to see that in the case of lines these are all, and that in the case of conics, these are all if we are given sufficiently good conditions. Also, there is more than one model $X \rightarrow \mathbb{P}^2$ to see X as a blow-up of the plane in 5 points. We show how we can choose one adequate to our needs.

Lemma 3.1. *The 16 lines in Table 1 are all the lines in X . The intersection of these lines are:*

$$\begin{aligned} E_i \cdot E_j &= -\delta_{ij}, & L_{ij} \cdot E_i &= L_{ij} \cdot E_j = 1, & C_0 \cdot E_i &= 1 & C_0^2 &= -1 \\ C_0 \cdot L_{ij} &= 1, & L_{ij} \cdot L_{kl} &= \begin{cases} -1 & \text{if } i = k \text{ and } j = l \\ 0 & \text{if only two subindices are equal} \\ 1 & \text{if none of the subindices are equal.} \end{cases} \end{aligned}$$

Lemma 3.2. *Given a line $L \subset X$, we can choose a model $\gamma: X \rightarrow \mathbb{P}^2$ such that $L = E_1$. If $p = L_1 \cap L_2$, the intersection of two lines, we can choose γ such that $L_1 = E_1$, $L_2 = E_2$.*

Proof. We construct $\gamma: X \rightarrow \mathbb{P}^2$ by contracting 5 disjoint exceptional curves F_i (i.e. $F_i \cdot F_j = 0$ if $i \neq j$). Let $F_1 = L$.

- (i) If $F_1 = E_1$, take $F_2 = E_2, F_3 = L_{34}, F_4 = L_{35}, F_5 = L_{45}$.
- (ii) If $F_1 = C_0$, take $F_j = L_{1j}$.
- (iii) If $F_1 = L_{12}$, take $F_i = L_{1(i+1)}$ for $2 \leq i \leq 4$, $F_5 = C_0$.

Obvious relabelling exhausts all 16 lines in Lemma 3.1. By Castelnuovo contractibility criterion [Har77, V.5.7] we can contract each F_i , leaving every other point intact. The image of γ is \mathbb{P}^2 , because the relative minimal model of X once 5 exceptional curves are contracted is unique. For the second part we can assume already $L_1 = E_1$ and run this lemma again. In that case we are in case (i) above and we are done. \square

In a similar fashion to Lemma 3.1 we can show:

Lemma 3.3. *If C is an irreducible conic in X passing through p , then $C = H_i$ or $C = B_i$, with $\pi(C)$ either a conic through all marked points but p_i or a line through p and p_i , respectively.*

Lemma 3.4. *Given C an irreducible conic in X , $p \in C$, we can choose a model $\gamma: X \rightarrow \mathbb{P}^2$ such that $C = H_i$ for any i , unless $p \in E_1$ in which case $i \neq 1$.*

Proof. If $p \in L$, a line in X , assume $L = E_1$ by Lemma 3.2. We have $C \neq H_1$ since otherwise

$$0 = H_1 \cdot E_1 = C \cdot E_1 \geq \text{mult}_p(C) \cdot \text{mult}_p(E_1) = 1,$$

a contradiction.

If $C = B_1$, take F_i and $\gamma : X \rightarrow \mathbb{P}^2$ as in Lemma 3.2 (i). Because C is irreducible, $\overline{C} = \gamma(B_1) = \mathcal{O}_{\mathbb{P}^2}(d)$ by the genus formula on \mathbb{P}^2 . Moreover:

$$B_1 \sim \gamma^*(\mathcal{O}_{\mathbb{P}^2}(d)) - \sum_{i=1}^5 (F_i \cdot B_1)^2 F_i = \pi^*(\mathcal{O}_{\mathbb{P}^2}(d)) - F_1 - F_3 - F_4 - F_5,$$

and $2 = B_1 \cdot (-K_X) = 3d - 4$, so $d = 2$. Therefore under the new blow-up C is H_2 . By obvious relabelling of the F_j we can consider $C = H_i$ with $i \neq 1$.

If $C = B_i$, with $i \neq 1$, then $p \notin E_1$ since C is irreducible. If $C = B_2$, the same choice of F_i gives us $C = H_1$ under the new blow-up. If $C = B_i$ for $i = 3, 4, 5$ take $F_1 = E_1, F_2 = E_i, F_3 = L_{jk}, F_4 = L_{jl}, F_5 = L_{kl}$ for different $j, k, l \in \{1, \dots, 5\} \setminus \{i\}$ and $C = H_i$ under γ . \square

3.3. Cubics in X .

Lemma 3.5. *Let \mathcal{LS} be a complete linear system of degree 3 as in the last three rows of Table 1. Then $\exists C' \in \mathcal{LS}$, a curve with $p \in C'$ satisfying one of the following:*

- (i) C' is smooth at p and its strict transform $\tilde{C}' \sim \sigma^*(C) - E$ passes through q .
- (ii) C' is reducible and two of its components intersect at p . One of this components is a line. By Lemma 3.2 we can assume this line is E_1 and either:
 - (a) $C' = E_1 + C$ for C an irreducible conic in Table 1 passing through p .
 - (b) $C' = E_1 + L_{12} + L$ for L a line not passing through p .

Case (ii)(a) is possible only if $\mathcal{LS} = \mathcal{R}_{ijk}$ for $(i, j, k) \in \{(2, 3, 4), (2, 4, 5), (3, 4, 5)\}$ or $\mathcal{LS} = \mathcal{R}$. Case (ii)(b) is possible only if $\mathcal{LS} = \mathcal{R}, \mathcal{Q}_2, \mathcal{R}_{2jk}, \mathcal{R}_{12k}$.

In case (ii) (a) we can find C :

- If $\mathcal{LS} = \mathcal{R}$, then $C = B_1$.
- If $\mathcal{LS} = \mathcal{R}_{234}$, then $C = H_5$.
- If $\mathcal{LS} = \mathcal{R}_{245}$, then $C = H_3$.
- If $\mathcal{LS} = \mathcal{R}_{345}$, then $C = H_2$.

In case (ii) (b) we can find L :

- If $\mathcal{LS} = \mathcal{R}$, then $L = E_2$.
- If $\mathcal{LS} = \mathcal{Q}_2$, then $L = C_0$.
- If $\mathcal{LS} = \mathcal{R}_{2jk}$, then $L = L_{jk}$.
- If $\mathcal{LS} = \mathcal{R}_{12k}$, then $L = L_{1k}$.

In each case, denote C' by the letter in the last column of Table 1. Note that in case (i), C' may still be reducible, but it is irreducible around p .

Proof. Let $\mathcal{LS}' = \{D \in \mathcal{LS} : p \in \text{Supp}(D)\}$ and let $\widetilde{\mathcal{LS}'} = |\sigma^*(\mathcal{LS}') - E|$. By Proposition 2.13

$$h^0(\widetilde{\mathcal{LS}'}) = h^0(\mathcal{LS}') = h^0(\mathcal{LS}) - 1 \geq 2,$$

so we can choose $B \in \widetilde{\mathcal{LS}'}$ an effective divisor passing through q . If $E \not\subset \text{Supp}(B)$, then let $C' = \sigma_*(B)$ and $B = \tilde{C}' \sim \sigma^*(C') - E$ where $B \cdot E = 1$. Clearly, this is case (i) in the statement.

Conversely, if $E \subset \text{Supp}(B)$, let $B = A + E$ where $E \not\subset \text{Supp}(A)$ and A is effective. Then $\tilde{C}' = A = B - E = \sigma^*(C') - 2E$ for $C' = \sigma_*(B) = \sigma_*(A)$ and C' is singular at p . C' is reducible, since otherwise $p_a(\tilde{C}') < p_a(C') = 0$, which is

impossible. This is case (ii) in the statement which can only split in subcases (a) and (b). In case (b) we can assume the second line is L_{12} by Lemma 3.2.

We prove (a). If $C' \in \mathcal{R}$, then $C = \pi^*(\mathcal{O}_{\mathbb{P}^2}(1)) - E_1 = B_1$. If $\mathcal{RS} = \mathcal{R}_{ijk}$, then $C = \pi^*(\mathcal{O}_{\mathbb{P}^2}(2)) - E_i - E_j - E_k - E_1$ which is not an irreducible conic in Lemma 3.3 unless it is one of the cases in the statement.

We prove (b). If $\mathcal{LS} = \mathcal{R}$, then $L = \pi^*(\mathcal{O}_{\mathbb{P}^2}(1)) - E_1 - L_{12} = E_2$. If $\mathcal{LS} = Q_i$, then

$$L = \pi^*(\mathcal{O}_{\mathbb{P}^2}(3)) - E_i - \sum_{j=1}^5 E_j - E_1 - L_{12} = \pi^*(\mathcal{O}_{\mathbb{P}^2}(2)) + E_2 - E_i - \sum_{j=1}^5 E_j,$$

which is not a line in Lemma 3.1 unless $i = 2$, in which case $L = C_0$. Finally, if $\mathcal{LS} = R_{ijk}$, then

$$L = \pi^*(\mathcal{O}_{\mathbb{P}^2}(2)) - E_i - E_j - E_k - E_1 - L_{12} = \pi^*(\mathcal{O}_{\mathbb{P}^2}(1)) - E_i - E_j - E_k + E_2$$

which is not a line unless $i = 1, j = 2, 3 \leq k \leq 5$ or $i = 2, 3 \leq j < k \leq 5$, finishing the proof. \square

3.4. Proof of the theorem. We aim to prove Theorem (1.6) in the case $K_X^2 = 4$.

Claim 3.6.

$$\text{lct}(X) \leq \frac{2}{3}$$

Proof. Take $p = E_1 \cap L_{12}$ and the conic H_2 , which is irreducible (see case 3 in section 3.5) and isomorphic to \mathbb{P}^1 . Consider $G = E_1 + L_{12} + H_2 \sim K_X$. We are done, since $\text{lct}_p(X, G) = 2/3$, and

$$\text{lct}(X) = \inf\{\text{lct}_r(X, D) : r \in X, D \sim_{\mathbb{Q}} K_X\} \leq \frac{2}{3}.$$

\square

We need to show $\text{lct}(X) \geq \frac{2}{3}$. We proceed by contradiction. Suppose there is an effective \mathbb{Q} -divisor

$$D = \sum d_i D_i \equiv -K_X, \quad d_i > 0 \ \forall i$$

such that λD is not log canonical for some $\lambda < \frac{2}{3}$. Then $\text{LCS}(X, \lambda D) \neq \emptyset$. We need an auxiliary result, which is straight forward from Bertini's Theorem when $k = \mathbb{C}$.

Lemma 3.7. *Given a curve $C \subset X$ with $\deg C \leq 2$, there is an irreducible curve Z such that $Z + C$ is cut out by a general hyperplane section of $X \subset \mathbb{P}^4$.*

Proof. Given $p \in C$, denote by $\sigma: \tilde{X} \rightarrow X$ the blowup at p and \tilde{C} the strict transform of C .

If $\deg C = 1$ we can assume $C = E_1$ by Lemma 3.2. Consider

$$Q_1 = |\pi^*(\mathcal{O}_{\mathbb{P}^2}(3)) - 2E_1 - E_2 - \dots - E_5|.$$

Choose $p = E_1 \cap L_{12}$ and $q = \sigma^{-1}(p) \cap \widetilde{E_1} \in \tilde{X}$. This defines a curve $Z = Q_1 \in \mathcal{Q}_1$ as in section 3.1 which is irreducible by the analysis in section 3.5, subcase 3.3.

If $\deg C = 2$ by Lemma 3.4 assume $C = H_1$. Choose $p \in H_1$ such that p is not in any line and take $Z = B_1 \in \mathcal{B}_1$ as in section 3.1. Z is irreducible by the analysis in section 3.5, subcase 1.

In both cases we have

$$C + Z \sim -K_X.$$

\square

Lemma 3.8. $\text{LCS}(X, \lambda D)$ contains only isolated points.

Proof. If $C \subset \text{LCS}(X, \lambda D)$, where C is a curve, then $C = D_i$ for some D_i such that $\lambda d_i \geq 1$, by Lemma 2.5 i.e. $d_i > \frac{3}{2}$. Then

$$4 = -K_X \cdot D = \sum d_i \deg(D_i) > \frac{3}{2} \deg(D_i),$$

so $\deg(D_i) \leq 2$. Using Lemma 3.7 choose a curve Z such that $D_i + Z$ is cut out by a general hyperplane section of X passing through D_i such that Z is irreducible. We have $D_i + Z \sim_{\mathbb{Q}} -K_X \equiv D$. Hence

$$(5) \quad \deg Z = Z \cdot D = (-K_X + D_i)(-K_X) = 4 - \deg D_i.$$

In particular $\deg Z \geq 2$, so $Z \cdot D_j \geq 0$ for all irreducible D_j (since only lines can have negative self-intersection). Then

$$(6) \quad (Z \cdot D) \geq d_i(Z \cdot D_i) = d_i(-K_X - D_i) \cdot D_i = d_i(\deg D_i - (\deg D_i - 2)) = 2d_i$$

by the genus formula, since all lines and conics in X are rational. But then (5) and (6) give a contradiction:

$$3 \geq 4 - \deg(D_i) \geq 2d_i > 3.$$

□

Consider the blow up of X at p , $\sigma : \tilde{X} \rightarrow X$ with exceptional divisor E . Applying Corollary 2.7 to λD , we obtain, that for some $q \in E$:

$$(7) \quad \text{mult}_q(\tilde{D}) + \text{mult}_p D > 3.$$

The following lemma is proved in section 3.5:

Lemma 3.9. Let $p \in X$, $q \in \tilde{X}$, the blowup of X at p . There is $H = \sum j_i J_i$, an effective \mathbb{Q} -divisor in X , with $H \sim_{\mathbb{Q}} -K_X$, such that:

- (i) $\frac{2}{3}H$ is log canonical.
- (ii) $p \in J_i \forall J_i$.
- (iii) $\deg J_i \leq 3 \forall J_i$.
- (iv) All J_i are irreducible.
- (v) $q \in \tilde{J}_i$, the strict transform of J_i via σ , $\forall J_i$ such that $\deg J_i > 1$.

Note that condition (i) can be replaced by $\text{mult}_p H \leq \frac{2}{3}$, by Lemma 2.5 (i).

By Lemma 2.4 we may assume that $\exists J_i$ such that $J_i \not\subset \text{Supp}(D)$.

Claim 3.10. If $\deg J_i = 1$, then $J_i \subset \text{Supp}(D)$.

Proof. Suppose $J_i \not\subset \text{Supp}(D)$. Then

$$\frac{3}{2} \leq \text{mult}_p(D) \leq \text{mult}_p(D) \cdot \text{mult}_p(J_i) \leq D \cdot J_i = (-K_X) \cdot J_i = 1$$

which is a contradiction. □

Hence, we can assume $\exists J_j \not\subset \text{Supp}(D)$ and $q \in \tilde{J}_j$, $p \in J_j$ and $2 \leq \deg(J_j) \leq 3$. Then

$$\tilde{J}_j \cdot \tilde{D} = J_j \cdot D - \text{mult}_p(J_j) \text{mult}_p(D) \leq 3 - \text{mult}_p(D).$$

But $\tilde{J}_j \not\subset \text{Supp}(\tilde{D})$, so

$$3 - \text{mult}_p(D) \geq \tilde{J}_j \cdot \tilde{D} \geq \text{mult}_q(\tilde{D}),$$

contradicting (7). This completes the proof.

3.5. Auxiliary divisors. In the proof in the previous section we used the existence of the effective \mathbb{Q} -divisors $H \sim_{\mathbb{Q}} -K_X$ satisfying certain conditions (see lemma 3.9). In this section we prove those lemmas by constructing these divisors explicitly, by case analysis, depending on the position of p and q . In order to do this we use curves from Table 1. These were constructed depending on points $p \in X$ and $q \in E$ and were possibly reducible. Conditions (ii) and (iii) in lemma 3.9 will be therefore clear by construction, as well as condition (v). We will check log canonicity (condition (i)) or use the shortcut $\text{mult}_p(H) \leq \frac{2}{3}$. The biggest task will be to prove that the curves chosen for each particular case are irreducible in each situation (condition (iv)).

Case 1.

Assumption 1: p is not in any line. In particular $p \notin E_i$ for all i .

Subcase 1.1.

Assumption 1.1: q is not in the strict transform of conics in X passing through p . Let $H = \frac{1}{2}R + \frac{1}{6}\sum_{i=1}^5 Q_i \sim_{\mathbb{Q}} -K_X$. Again, Assumption 1 and Assumption 1.1 assure that R and Q_i are irreducible, so we just need to check

$$2\text{mult}_p(H) = 2 \left[\frac{1}{2} + 5 \cdot \frac{1}{6} \right] = \frac{16}{6} < 3.$$

Subcase 1.2.

Assumption 1.2: $q \notin \tilde{L}$, for L a line in X , but $q \in \tilde{C}$, the strict transform of a conic in X . Without loss of generality assume $C = H_1$, which is irreducible (see Lemma 3.4). Let

$$H = \frac{1}{2}H_1 + \frac{1}{2}R_{125} + \frac{1}{2}R_{134} \sim_{\mathbb{Q}} -K_X.$$

$q \in \tilde{R}_{125} \cap \tilde{R}_{124}$ by Lemma 3.5, since p does not lie in a line. We need to show R_{125} and R_{134} are irreducible. By relabelling, it is enough to show it for R_{125} . Suppose $R_{125} = C_a + L_b$ where C_a is a conic and L_b is a line. $p \in C_a$ and $p \notin L_b$, by Assumption 1. Therefore, by construction $q \in \tilde{C}_a$, and C_a is irreducible, by Assumption 1.2. But then, by Lemma 3.3 and Lemma 3.4, $C_a = H_i$, where $i = 3, 4$. Suppose without loss of generality that $i = 3$. Then $L_b = R_{125} - H_3 = E_4$, and

$$(8) \quad 4 = \pi(H_3) \cdot \pi(H_1) = \sum_{r \in \pi(H_3) \cap \pi(H_1)} \text{mult}_r(\pi(H_3) \cap \pi(H_1)).$$

Since $p \notin E_k$ and $p \in H_3 \cap H_1$, we have that $\pi(H_3) \cap \pi(H_1) = \{p_2, p_4, p_5, \pi(p)\}$, with $\pi(p) \neq p_j$ for all j , by Assumption 1. Hence, (8) implies

$$\text{mult}_p(\pi(H_3) \cap \pi(H_1)) = 1.$$

In this case H_3 and H_1 intersect transversally at p so $q \notin \tilde{H}_3 = \tilde{C}_a$, contradicting the assertion that R_{125} is reducible. Finally

$$2\text{mult}_p(H) = 2 \left[\frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right] = 3.$$

Case 2. Suppose $p \in L$, a line in X and no other line. By Lemma 3.2 we can consider $L = E_1$.

Assumption 2: $p \in E_1$ and $p \notin L$, any other line different than E_1 .

Subcase 2.1.

Assumption 2.1: $q \notin \tilde{C}$ for C any line or conic in X . In particular $q \notin \tilde{E}_1$. Let

$$H = \frac{1}{8} \sum_{2 \leq j < k \leq 5} R_{1jk} + \frac{1}{8} \sum_{i=2}^5 Q_i + \frac{1}{4} E_1 \sim_{\mathbb{Q}} -K_X$$

$q \in \tilde{Q}_i, \tilde{R}_{1jk}$ by Lemma 3.5 since p does not lie in two lines. All R_{1jk} and Q_i are irreducible, since otherwise they would split in a conic C_a and a line L_b and either $q \in \tilde{C}_a$ or $q \in \tilde{L}_b$, contradicting the assumption. Moreover

$$2(\text{mult}_p(H)) = 2 \left(\frac{1}{8} \cdot 6 + \frac{1}{8} \cdot 4 + \frac{1}{4} \cdot 1 \right) = 3.$$

Subcase 2.2.

Assumption 2.2: $q \in \tilde{C}$, for some conic C in X but $q \notin \tilde{L}$, for all lines L in X . In particular C is irreducible and $q \notin \tilde{E}_1$. By lemmas 3.3 and 3.4 we can assume that

$$C = \pi^*(\mathcal{O}_{\mathbb{P}^2}(2)) - \sum_{\substack{i=1 \\ i \neq k}}^5 E_i \sim_{\mathbb{Q}} H_k, \text{ for } k \neq 1.$$

where $p \in C = H_k$, $k \neq 1$, with $q \in \tilde{C}$. Moreover, since $q \in \tilde{H}_k$, Assumption 2.2 assures it is irreducible.

Without loss of generality, suppose $k = 5$. Suppose there is another conic C' in X such that $p \in C'$, $q \in \tilde{C}'$ and $C' \neq H_5$. Since $p \in C' \cap E_1$, by Lemma 3.3 either $C' = B_1$ or $C' = H_j$, for $j \neq 1, 5$. However $H_5 \cdot B_1 = 1$, $H_i \cdot H_5 = 1$ for $i \neq 1, 5$. Therefore in both cases C' and H_5 intersect transversally and $q \notin \tilde{C}'$. Let

$$(9) \quad H = \frac{3}{5} H_5 + \frac{1}{5} (R_{125} + R_{135} + R_{145}) + \frac{1}{5} Q_5 + \frac{2}{5} E_1 \sim_{\mathbb{Q}} -K_X.$$

$q \in \tilde{Q}_5, \tilde{R}_{1jk}$ by Lemma 3.5 since p does not lie in two lines.

We already know that H_5 and E_1 are irreducible. Suppose Q_5 is reducible. Then $Q_5 = H_5 + L_b$, where L_b is a line. But then

$$L_b \sim_{\mathbb{Q}} Q_5 - H_5 \sim \pi^*(\mathcal{O}_{\mathbb{P}^2}(1)) - 2E_5$$

which is not one of the lines in X , by Lemma 3.1.

If R_{125} is reducible, then there is a line L_a such that

$$L_a \sim_{\mathbb{Q}} R_{125} - H_5 \sim -E_5 + E_4 + E_3$$

which is not a line in Lemma 3.1. By relabelling, it is clear that R_{135} and R_{145} are irreducible too.

Lemma 3.11. *If $p \in H_5 \cap E_1$ and $q \in \tilde{H}_5$, $q \notin \tilde{L}$, for any line L in X , then $\frac{2}{3}H$ is log canonical, for H as in (9)*

Proof. Let $\sigma_0 : X_0 \rightarrow X$ be the blow up at p with exceptional divisor F_1 with $q \in F_1$. Table 2 gives the intersection numbers in X_0 . Since all curves in Table 2 intersect normally and pass through q , we just need to blow up this point to obtain

	\widetilde{H}_5	\widetilde{R}_{125}	\widetilde{R}_{135}	\widetilde{R}_{145}	\widetilde{Q}_5	\widetilde{E}_1	F_1
\widetilde{H}_5		1	1	1	1	0	1
\widetilde{R}_{125}			1	1	1	0	1
\widetilde{R}_{135}				1	1	0	1
\widetilde{R}_{145}					1	0	1
\widetilde{Q}_5						0	1
\widetilde{E}_1							1

TABLE 2. Intersection numbers for Lemma 3.11

snc. Let $\sigma : \widetilde{X} \rightarrow X$ be the composition of both blowing up maps and F_2 be the second exceptional divisor. Then:

$$\sigma^*(\lambda H + K_{\widetilde{X}}) = \lambda \widetilde{H} + \left(\left(\frac{3}{5} + 3 \cdot \frac{1}{5} + \frac{1}{5} + \frac{2}{5} \right) \lambda - 1 \right) F_1 + \left(\left(\frac{7}{5} + \frac{9}{5} \right) \lambda - 2 \right) F_2,$$

and for $\lambda = 2/3$, λH is log canonical. \square

Subcase 2.3.

Suppose that in Assumption 2 $q \in \widetilde{L}$ for some line L in X . Then we can assume $L = E_1$ by Lemma 3.2.

Assumption 2.3: $q \in \widetilde{E}_1$. Suppose $q \in \widetilde{C}$ where C is a conic in X . As in case 2.1 we can assume, by using Lemma 3.3 and Lemma 3.4 that $C = H_5 = \pi^*(\mathcal{O}_{\mathbb{P}^2}(2)) - \sum_{i=1}^4 E_i$. C is irreducible, since otherwise $L_a = H_5 - E_1 \sim_{\mathbb{Q}} \pi^*(\mathcal{O}_{\mathbb{P}^2}(2)) - 2E_1 - \sum_{i=2}^4 E_i$, would be a line.

Since C is irreducible, it intersects E_1 transversally at p and since $q \in \widetilde{E}_1$, then $q \notin \widetilde{C}$. So q does not belong to the strict transform of any conic. Now, take

$$H = Q_1 + E_1 \sim_{\mathbb{Q}} -K_X,$$

and $\frac{2}{3}H$ is log canonical. Since p is not the intersection of two lines, $q \in \widetilde{Q}_1$ by Lemma 3.5. Since q is not on the strict transform of any conic, if Q_1 is not irreducible, then $Q_1 = E_1 + C_a$, where C_a is a (possibly irreducible) conic such that $q \notin \widetilde{C}_a$, $p \in C_a$. But

$$C_a = Q_1 - E_1 \sim_{\mathbb{Q}} \pi^*(\mathcal{O}_{\mathbb{P}^2}(3)) - 3E_1 - \sum_{i=2}^5 E_i$$

is not an irreducible conic, by Lemma 3.3. If it was the union of two lines, by Lemma 3.1, one of them should be C_0 , but then

$$L_b = C_a - C_0 \sim_{\mathbb{Q}} \pi^*(\mathcal{O}_{\mathbb{P}^2}(1)) - 2E_1$$

is not one of the lines in Lemma 3.1.

Case 3.

Suppose $p = L_a \cap L_b$, the intersection of 2 lines. By Lemma 3.2 we can assume:

Assumption 3: $p = E_1 \cap L_{12}$.

Subcase 3.1.

Assumption 3.1: $q \notin \tilde{C}$ for $C \subset X$, a line or a conic in X . Let

$$H = Q_1 + E_1.$$

$q \in \tilde{Q}_1$ by inspection in Lemma 3.5. By assumption Q_1 is irreducible. $\frac{2}{3}H$ is clearly log canonical.

Subcase 3.2.

Assumption 3.2: $q \in \tilde{C}$ for $C \subset X$, a conic in X and $q \notin \tilde{L}$ for any $L \subset X$, a line in X . By assumption, C is irreducible. It is easy to see that $C = H_2$. In fact, if $C = H_i$ for $i \neq 1, 2$ or $C = B_1$, then $0 = C \cdot L_{12} \geq \text{mult}_p(C) \cdot \text{mult}_p(L) \geq 1$, a contradiction.

If $C = H_1$ or $C = B_i$, for $i \neq 1$, then $0 = C \cdot E_1 \geq \text{mult}_p C \cdot \text{mult}_p E_1 \geq 1$, a contradiction.

Let

$$H = H_2 + L_{12} + E_1 \sim -K_X.$$

H_2 is irreducible, since otherwise there would be a line L passing through q . Since

$$H_2 \cdot L_{12} = H_2 \cdot E_1 = L_{12} \cdot E_1 = 1,$$

then $\text{lct}(X, H) = \frac{2}{3}$.

Subcase 3.3. Suppose $q \in \tilde{L}$, the strict transform of a line in X . Then either $L = E_1$ or $L = L_{12}$. If $L = L_{12}$ by Lemma 3.2 we can choose another model $\gamma : X \rightarrow \mathbb{P}^2$ such that $F_1 = L_{12}$ is an exceptional divisor. Then this case is essentially 2.3. Using the same argument as in that case, there is no conic C through p such that $q \in \tilde{C}$. $q \in \tilde{Q}_1$ by Lemma 3.5 and Q_1 is irreducible, hence we take

$$H = Q_1 + E_1,$$

and again $\frac{2}{3}H$ is log canonical.

4. DEL PEZZO SURFACE OF DEGREE 2

Let X be a del Pezzo surface of degree 2. We prove Theorem 1.6 for this case. Let $\omega = \frac{3}{4}$ if $|-K_X|$ has a tacnodal curve and $\omega = \frac{5}{6}$ otherwise.

Claim 4.1.

$$\text{lct}(X) \leq \omega$$

Proof. By [Par01], if $\exists C \in |K_X|$ a tacnodal curve, then $\text{lct}(X, C) = \frac{3}{4}$. Otherwise we can take $C \in |K_X|$ a cuspidal rational curve and $\text{lct}(X, C) = \frac{5}{6}$. \square

Claim 4.2.

$$\text{lct}(X) \geq \omega$$

Proof. Suppose there is $\lambda < \omega < 1$ such that $(X, \lambda D)$ is not log canonical for D an effective \mathbb{Q} -divisor, $D \sim_{\mathbb{Q}} -K_X$. Proceeding as in Lemma 3.8 we see that the locus of log canonical singularities is just made up of isolated points. Let $p \in X$ be one of these points. Let $\mathcal{C} \subset |-K_X|$ be the sublinear system fixing p . Suppose $\exists C \in \mathcal{C}$ singular at p . $(X, \lambda C)$ is log canonical, so by Lemma 2.4 there is one component of C not in $\text{Supp}(D)$. C is reducible, since otherwise:

$$2 = C \cdot D \geq \text{mult}_p D \text{ mult}_p C \geq \frac{2}{\lambda} > 2.$$

If $C = L_1 + L_2$, the union of two lines intersecting at p , by convexity we may assume that $L_1 \not\subset \text{Supp}(D)$. Then

$$(10) \quad 1 = L_1 \cdot D \geq \text{mult}_p L_1 \cdot \text{mult}_p D > \frac{1}{\omega} > 1,$$

a contradiction.

We may therefore assume that all $C \in |-K_X|$ passing through p are non-singular at p . Let $\sigma : \tilde{X} \rightarrow X$ be the blow-up at p with exceptional divisor E and by Proposition 2.13 pick $C \in \mathcal{C}$ such that $q \in \tilde{C}$. By convexity, if C is irreducible, then $C \not\subset \text{Supp}(D)$ and by

$$2 - \text{mult}_p D = \tilde{C} \cdot \tilde{D} \geq \text{mult}_q(\tilde{D}) > \frac{2}{\omega} - \text{mult}_p(D),$$

we obtain a contradiction. Hence $C = L_1 + L_2$, the union of two lines $p \in L_1$, $p \notin L_2$. The intersection numbers are:

$$L_1 \cdot L_2 = 2, \quad L_1^2 = L_2^2 = -1.$$

Since $(X, \omega C)$ is log canonical, by convexity we can assume $L_1 \subset \text{Supp}(D)$ since otherwise the computation in (10) gives a contradiction. Therefore we may write $D = mL_1 + \Omega$ with $m > 0$, $L_1, L_2 \not\subset \text{Supp}(\Omega)$. Then

$$1 = L_2 \cdot D = 2m + L_2 \cdot \Omega \geq 2m,$$

so $m \geq \frac{1}{2}$. On the other hand, since $q \in \tilde{L}_1$, we obtain:

$$2 - \text{mult}_p D \geq 1 - \text{mult}_p D + 2m = \tilde{L}_1 \cdot (\tilde{D} - m\tilde{L}_1) = \tilde{L}_1 \cdot \tilde{\Omega} > \frac{2}{\omega} - \text{mult}_p D,$$

which is impossible. \square

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